LOW-FREQUENCY STABLE ELECTRO-QUASISTATIC FIELD FORMULATIONS BASED ON PENALTY APPROXIMATIONS OF CONTINUOUS EXTENSIONS

F. Kasolis, M.-L. Henkel, M. Clemens

Chair of Electromagnetic Theory, University of Wuppertal, 42119 Wuppertal, Germany
{kasolis, mhenkel, clemens}@uni-wuppertal.de

Keywords: Continuous extensions, electro-quasistatic fields, finite-element method, low-frequency stabilization.

Abstract

A framework for stabilizing the low-frequency instability that arises in electro-quasistatic field simulations, under the presence of non-conducting material, is developed. The resulting symmetric formulations rely on penalization for imposing the electric Gauß law in void, and hence, they constitute approximations of continuously extended problems. Real and imaginary penalty-weights are studied numerically, in terms of accuracy and conditioning.

1 Introduction

Electro-quasistatic (EQS) fields-models [1] encapsulate resistive and capacitive phenomena at low frequencies. The validity regime of EQS field-models includes high-voltage components, such as, cables, long-rod insulators, and surge arresters. In a numerical setting [2], the discrete EQS operators become increasingly badly-conditioned with decreasing frequency for domains that contain non-conductive material, and hence, they result in low-frequency (LF) instabilities that can be tracked down to the lack of uniqueness of EQS fields in the static limit $\omega = 0$. To cure the resulting numerical instability, various stabilization strategies have been proposed, such as, the generating system approach [3]. Here, a LF-stabilization framework that is based on penalizing the electric Gauß law in void is documented and its performance is numerically assessed.

To introduce the setting of a typical EQS field problem, consider an open, bounded, and simply-connected domain Ω whose boundary is Lipschitz, such as the domain that is depicted in Fig. 1. Suppose that Ω is free from sources and it constitutes of two subdomains Ω_V and Ω_C whose constant conductivities are $\sigma_\mathrm{V} = 0$ and $\sigma_\mathrm{C} > 0$, respectively. The time-harmonic EQS problem of interest is

$$\nabla \cdot (\kappa \nabla \varphi) = 0 \text{ in } \Omega, \tag{1}$$

$$\varphi | \Gamma_\mathrm{G} = 0,\ \varphi | \Gamma_\mathrm{S} = \varphi_\mathrm{S},\ \partial_n \varphi | \Gamma_\mathrm{I} = 0, \tag{2}$$

where φ is the sought scalar EQS potential and $\kappa = \sigma + \mathrm{i}\omega\varepsilon$, with σ being the electrical conductivity, i being the imaginary unit, $\omega = 2\pi f$ being the angular frequency, and ε being the subdomain-wise constant permittivity. The boundary Γ_G is grounded, while Γ_S supplies potential $\varphi_\mathrm{S} > 0$, and Γ_I is insulating. Provided that $(\cdot,\cdot)_\Omega$ denotes the standard inner product on $L^2(\Omega)$, the variational form of EQS problem (1), (2) reads: find $\varphi \in H^1(\Omega)$ such that $\varphi|\Gamma_\mathrm{G} = 0$, $\varphi|\Gamma_\mathrm{S} = \varphi_\mathrm{S}$, and

$$(\mathrm{i}\omega\varepsilon_0 \nabla\varphi, \nabla\psi)_{\Omega_\mathrm{V}} + (\kappa\nabla\varphi, \nabla\psi)_{\Omega_\mathrm{C}} = 0 \tag{3}$$

for all $\psi \in H^1(\Omega)$ that vanish on Γ_G and Γ_S. Although well-posedness of EQS problem (3) follows from the Lax-Milgram theorem for all real $\omega > 0$, the ω-weighted volume integral over Ω_V is nearly vanishing in the LF-regime and the resulting discrete operators are expected to be badly-conditioned.

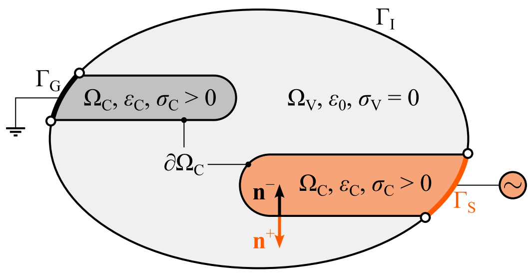

Figure 1. Conceptual setting for problem (1), (2), where Ω_V is void and Ω_C is occupied by conducting material.

2 Continuous Extension and Stabilization

Continuous extensions have been introduced in various contexts, such as in domain decomposition [4] and in fictitious-domain methods [5, 6]. Here, the EQS data on the interior conducting interfaces are continuously extended in void, with the electric Gauß law, for stabilizing the EQS problem in the LF-regime. To briefly justify the LF-stabilized EQS problem below, choose test functions that vanish in Ω_C and observe that equation (3) takes the form $(\varepsilon_0\nabla\varphi, \nabla\psi)_{\Omega_\mathrm{V}} = 0$ for all ψ that vanish in $\Omega_\mathrm{C} \cup \partial\Omega_\mathrm{C}$ and all positive frequencies, while

$$\omega = 0 \Rightarrow (\varepsilon_0\nabla\varphi, \nabla\psi)_{\Omega_\mathrm{V}} = 0, \tag{4}$$

according to the electric Gauß law $\nabla \cdot (\varepsilon_0 \nabla\varphi) = 0$. Hence, the EQS potential can be continuously extended from the conducting interface $\partial\Omega_\mathrm{C}$ into Ω_V, with the boundary value problem: find $\varphi_\mathrm{V} \in H^1(\Omega_\mathrm{V})$ such that $\varphi_\mathrm{V}|\partial\Omega_\mathrm{C} = \varphi|\partial\Omega_\mathrm{C}$ and

$$(\varepsilon_0\nabla\varphi_\mathrm{V}, \nabla\psi)_{\Omega_\mathrm{V}} = 0 \tag{5}$$

For all $\psi \in H^1(\Omega)$ that vanish on $\partial\Omega_\mathrm{C}$. By relaxing the test functions in $H^1(\Omega_\mathrm{V})$ and adding an α-multiple of (5) to (3), with complex $\alpha \neq 0$, the LF-stabilized equation

$$\left((\mathrm{i}\omega + \alpha)\varepsilon_0\nabla\phi, \nabla\psi\right)_{\Omega_\mathrm{V}} + (\kappa\nabla\phi, \nabla\psi)_{\Omega_\mathrm{C}} = 0 \tag{6}$$

is obtained. The solution to problem (6) is expected to be proximal to that of problem (3), with differences that emerge from the fact that the relaxation of the test functions yields $(\varepsilon_0\nabla\varphi_\mathrm{V}, \nabla\psi)_{\Omega_\mathrm{V}} = (\varepsilon_0\partial_n\varphi_\mathrm{V}, \psi)_{\partial\Omega_\mathrm{C}}$. It is worth pointing out that for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Im}(\alpha) = 0$, the LF-stabilization introduces an artificial conductivity $\sigma_\mathrm{A} = \alpha\varepsilon_0$ in void, while the choice $\mathrm{Re}(\alpha) = 0$ and $\mathrm{Im}(\alpha) > 0$ modifies the frequency in void according to $\omega|\Omega_\mathrm{V} = \omega + |\alpha|$.

3 Numerical Experiments

The accuracy and the conditioning of the LF-stabilized EQS problem are studied for the three-dimensional test-case capacitor that is depicted in Fig. 2, where the values $\varepsilon_\mathrm{C} = \varepsilon_0 \cong 8.854 \cdot 10^{-12}$ F/m, $\sigma_\mathrm{C} = 10^6$ S/m, and $\varphi_\mathrm{S} = 1$ V are used.

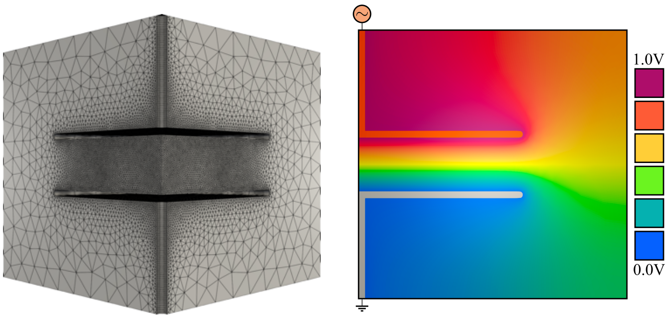

Figure 2. The computational mesh and the real part of the potential that is obtained with problem (3) for $f = 50$ Hz.

To perform numerical experiments, a three-dimensional mesh is generated by a $\pi/2$-rotation of a two-dimensional mesh, using ten layers. The discrete finite-element operators are assembled with first-order Lagrangian elements and the resulting number of degrees of freedom is approximately $3 \cdot 10^5$. Accuracy is assessed by solving the linear systems that are associated with both problems (3) and (6), using a direct solver, and afterwards, computing the relative differences

$$e_p(\alpha) = \|p(\varphi) - p(\phi)\|_\Omega / \|p(\varphi)\|_\Omega, \tag{7}$$

where $p \in \{\mathrm{Re}, \mathrm{Im}\}$ and $\|\cdot\|_\Omega$ is the standard $L^2(\Omega)$-norm. In Fig. 3, $e_p(\alpha)$ is plotted for $\alpha \in \{10^0, 10^1, \dots, 10^{10}\}$, using $\log$- $\log$ scale. The conditioning is assessed by estimating the condition number, see Fig. 4. In the full paper, similar experiments will be presented for imaginary α-values, for various conductors, for a wider range of frequencies, and for more suitable test-problems, whenever possible.

Acknowledgements

This work was supported in parts by the German Research Foundation (DFG) under grant no. CL143/18-1.

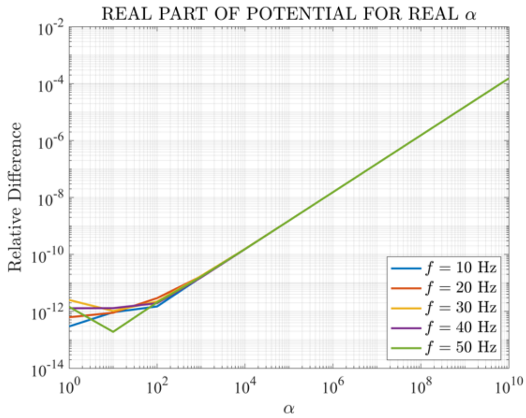

Figure 3. The relative differences $e_\mathrm{Re}(\alpha)$ for the frequency-values that are mentioned at the legend.

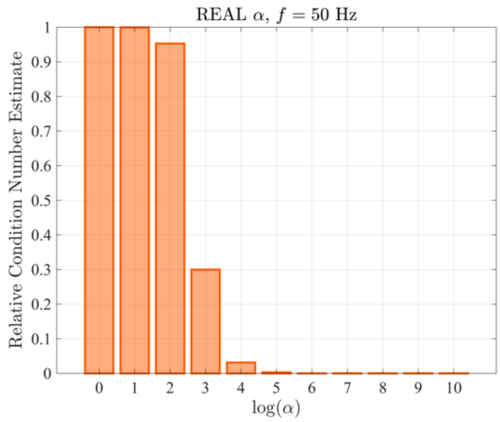

Figure 4. An estimate of the condition number relative to the maximum condition number.

References

[1] H. K. Dirks, “Quasi-Stationary Fields for Microelectronic Applications”, *Electr. Eng.*, vol. 79, pp. 145–155, 1996.

[2] L. Egiziano, V. Tucci, C. Petrarca, M. Vitelli, “A Galerkin Model to Study the Field Distribution in Electrical Components Employing Nonlinear Stress Grading Materials”, *IEEE Trans. Dielectr. Electr. Insul.*, vol. 6, no. 6, pp. 765–773, 1999.

[3] R. Hiptmair, F. Kramer, J. Ostrowski, “A Robust Maxwell Formulation for All Frequencies”, *IEEE Trans. Mag.*, vol. 44, no. 6, pp. 682–685, 2008.

[4] V. Dolean, P. Jolivet, F. Nataf, “An Introduction to Domain Decomposition Methods: Algorithms, Theory, and Parallel Implementation”, *SIAM*, 2016.

[5] R. Glowinski, T.-W. Pan, “Error Estimates for Fictitious Domain / Penalty / Finite Element Methods”, *Calcolo*, vol. 29, pp. 125–141, 1992.

[6] F. Kasolis, E. Wadbro, M. Berggren, “Analysis of Fictitious Domain Approximations of Hard Scatterers”, *SIAM J. Numer. Anal.*, vol 5, pp. 2347–2362, 2015.