Port-Hamiltonian System Framework for Conservatively Coupled Discrete Electromagnetics and Multi-Physics Problem Formulations

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Abstract

The port-Hamiltonian (pH) framework gives rise to mathematical models that preserve physical quantities, such as the energy, or maintain dissipation inequalities. Within multiphysics problems, submodels that appear as port-Hamiltonian systems (pHS) with input/output quantities that are coupled linearly, for many applications with skew-symmetric matrices, result in global pH systems. In this contribution, the suitability of the pH framework is demonstrated for systems of equations that result from mimetic discretizations of electromagnetism, such as the finite-integration technique (FIT), for network formulations, and for thermodynamic field formulations.

1 Introduction

Current electromagnetic and electronic devices often have geometric multiscale and temporal multirate characteristics that result in complex and computationally infeasible full Maxwell models, in a practical setting. Such devices can often be decomposed into multiple and multiply coupled submodels, such as for instance, field-circuit and field-transmission line models. In addition, electromagnetic phenomena can result in substantial thermal and mechanical effects, and hence, a complete description requires a multi-physical approach.

Port-Hamiltonian system formulations [1] that either appear in a continuous setting (as sets of partial differential equations) or in discrete variants (as pH differential-algebraic equations, pH-DAE), provide a framework for coupled system formulations in multi-model and multi-physics problems. The resulting pHS formulations enable energy conservation and dissipation inequalities. Whenever these properties are satisfied in the pHS subsystems in coupled formulations and the input/output variables are linearly related, also the coupled system is pHS [2]. For instance, with $i = 1,2,\dots,n$, autonomous pH-DAEs of the form

(1a) $$\mathbf{E}_i \frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = (\mathbf{J}_i - \mathbf{R}_i) z_i(\mathbf{x}_i) + \widehat{\mathbf{B}}_i \widehat{\mathbf{u}}_{i,\mathrm{int}}(t) + \overline{\mathbf{B}}_i \overline{\mathbf{u}}_i(t)$$

(1b) $$\widehat{\mathbf{y}}_i(t) = \widehat{\mathbf{B}}_i^\top z_i(\mathbf{x}_i),\ \overline{\mathbf{y}}_i(t) = \overline{\mathbf{B}}_i^\top z_i(\mathbf{x}_i),$$

with possibly nonlinear mappings z_i, skew symmetric matrices $\mathbf{J}_i = -\mathbf{J}_i^\top$, symmetric and semi-positive definite dissipation matrices $\mathbf{R}_i \geq 0$. The linear input/output relations (1b) split up into internal (hat) and external (bar) quantities and a linear coupling relation $\widehat{\mathbf{u}} + \mathbf{K}\widehat{\mathbf{y}} = \mathbf{0}$ of the internal quantities, with a skew-symmetric matrix $\mathbf{K} = -\mathbf{K}^\top$, as is often the case in applications. These latter systems can again be combined into a pH-DAE system

(2a) $$\mathbf{E}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = (\overline{\mathbf{J}} - \mathbf{R}) z(\mathbf{x}) + \overline{\mathbf{B}}\overline{\mathbf{u}}(t),$$

(2b) $$\overline{\mathbf{y}}(t) = \overline{\mathbf{B}}^\top z(\mathbf{x}),$$

with a skew-symmetric matrix $\overline{\mathbf{J}} = \mathbf{J} - \mathbf{B}\mathbf{K}\mathbf{B}^\top$ [4]. Various pHS formulations have been studied for individual and for coupled problems. For instance, pHS formulations for Maxwell equations are analyzed in [4], electric circuit problems in [1,4,5], coupled discrete field-circuit formulations in [3,6], both using the network-type equations of the finite integration technique [7]. Coupled circuit-transmission line pHS formulations are presented in [8,9].

2 Maxwell Grid Equations in pH-DAE form

The structure of the Maxwell equations already corresponds to a pHS formulation, which is mimicked within a pH-DAE for the discrete Maxwell grid equations of the Finite integration technique,

(3) $$\begin{bmatrix} \mathbf{M}_\varepsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_\mu \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{e} \\ \mathbf{h} \end{bmatrix} = \left(\underbrace{\begin{bmatrix} \mathbf{0} & \tilde{\mathbf{C}} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix}}_{\mathbf{J}} - \underbrace{\begin{bmatrix} \mathbf{M}_\kappa & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathbf{R}} \right) \begin{bmatrix} \mathbf{e} \\ \mathbf{h} \end{bmatrix} + \underbrace{\begin{bmatrix} -\mathbf{1} \\ \mathbf{0} \end{bmatrix}}_{\mathbf{B}} \mathbf{j}_s(t), \qquad \mathbf{y}(t) = \mathbf{B}^\top \begin{bmatrix} \mathbf{e} \\ \mathbf{h} \end{bmatrix},$$

where $\mathbf{e}(t)$ and $\mathbf{h}(t)$ are the (spatial) degrees of freedom (dof) vectors of electro- and magnetomotive forces assigned to the edges of the primal-dual grid complex, and $\mathbf{M}_\varepsilon$, $\mathbf{M}_\mu$, $\mathbf{M}_\kappa$ are diagonal approximations of the associated Hodge star permittivity, permeability, and electrical conductivity operators, respectively. Since $\mathbf{M}_\kappa \geq 0$, the corresponding matrix $\mathbf{R}$ is diagonal and semi-positive definite. The duality property $\mathbf{C}^\top = \tilde{\mathbf{C}}$ of the two discrete primal and dual grid curl operators, implies the skew-symmetry of $\mathbf{J} = -\mathbf{J}^\top$, and hence, (3) is a pH-DAE system with the corresponding stability and passivity results, already shown in [7] using an alternative argumentation. As the matrix $\mathbf{E}$ may also be singular, formulation (3) also includes a discrete magneto-quasistatic pH-DAE formulation with $-\dot{\mathbf{a}} = -\mathbf{d}\mathbf{a}/\mathrm{d}t = \mathbf{e}$, so that

(4)
$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_\mu \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} -\dot{\mathbf{a}} \\ \mathbf{h} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{0} & \tilde{\mathbf{C}} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{M}_\kappa & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} -\dot{\mathbf{a}} \\ \mathbf{h} \end{bmatrix} + \begin{bmatrix} -\mathbf{1} \\ \mathbf{0} \end{bmatrix} \mathbf{j}_s(t), \qquad \mathbf{y}(t) = \mathbf{B}^\top \begin{bmatrix} -\dot{\mathbf{a}} \\ \mathbf{h} \end{bmatrix}.$$

In the discrete electro-quasistatic pH-DAE reformulation of (3), the vector of magnetic grid voltages can be replaced by an electric vector potential $\mathbf{t}(t) = \mathbf{h}$, and yields

(5)
$$\begin{bmatrix} \mathbf{M}_\varepsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{e} \\ \mathbf{t} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{0} & \tilde{\mathbf{C}} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{M}_\kappa & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathbf{e} \\ \mathbf{t} \end{bmatrix} + \begin{bmatrix} -\mathbf{1} \\ \mathbf{0} \end{bmatrix} \mathbf{j}_s(t), \qquad \mathbf{y}(t) = \mathbf{B}^\top \begin{bmatrix} -\dot{\mathbf{a}} \\ \mathbf{h} \end{bmatrix}.$$

Applying e.g. an implicit Euler time-integration and a Schur complement, and with $\mathbf{M} = [(\Delta t)^{-1}\mathbf{M}_\varepsilon + \mathbf{M}_\kappa]^{-1}$, the pH-DAE system (5) yields a so far unpublished electro-quasistatic curl-curl formulation time stepping scheme,

(6)
$$\mathbf{e}^{n+1} = \mathbf{M}\big(\mathbf{I} - \tilde{\mathbf{C}}[\mathbf{C}\mathbf{M}\tilde{\mathbf{C}}]^{-\#}\mathbf{C}\mathbf{M}\big)\left(\tfrac{1}{\Delta t}\mathbf{M}_\varepsilon \mathbf{e}^n - \mathbf{j}_s^{n+1}\right).$$

2 Electric Circuit pH-DAE Formulation

For electric circuits, pH-DAE formulations [5] are established. A first pH-DAE field-circuit formulation is derived in [6], assuming ports on the terminals of circuits. An alternative coupled field-circuit pH-DAE formulation based on modified nodal analysis was recently presented in [4].

3 Transmission Line pH-DAE Formulation

For multiconductor transmission line formulations used for modelling transversal electromagnetic wave propagation along electric cable harnesses, pHS formulations are analyzed in [8,9]. A discrete transmission line model with the dof vector $[\mathbf{u}, \mathbf{i}]^T(t)$ of voltages and currents reads as

(7)
$$\begin{bmatrix} \mathbf{M}_C & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_L \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{u} \\ \mathbf{i} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{0} & \mathbf{P} \\ \tilde{\mathbf{P}} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{M}_G & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_R \end{bmatrix} \right) \begin{bmatrix} \mathbf{u} \\ \mathbf{i} \end{bmatrix} + \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{\mathrm{Rhs}} \\ \mathbf{u}_{\mathrm{Rhs}} \end{bmatrix}(t), \qquad \mathbf{y}(t) = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}^\top \begin{bmatrix} \mathbf{i} \\ \mathbf{u} \end{bmatrix}(t),$$

with symmetric positive (semi-)definite matrices $\mathbf{M}_C$, $\mathbf{M}_L$, $\mathbf{M}_R$, $\mathbf{M}_G$ corresponding to transmission line parameters. The pH-DAE formulation (7) relies on the duality relation of the one-dimensional difference operators $\tilde{\mathbf{P}} = -\mathbf{P}^\mathrm{T}$. Electromagnetic field-transmission line pH-DAE formulations for the description of scattering and radiation from electric transmission lines require the definition of port structures along the transmission line geometries.

4 Thermal pH-DAE Formulation

A FIT discretization of the heat equation [10] yields the ordinary differential equations (ODE) system

(8)
$$\mathbf{M}_{c\rho} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T} = -\tilde{\mathbf{S}}\mathbf{j}_w + \mathbf{q}_w(t), \;\; \mathbf{j}_w = -\mathbf{M}_\lambda \mathbf{G}\mathbf{T},$$

and $\mathbf{f}_w = \mathbf{M}_\lambda^{-1}\mathbf{j}_w$, where $\mathbf{T}(t)$ is the dof vector of the primal grid node temperatures, $\mathbf{M}_{c\rho}$ and $\mathbf{M}_\lambda$ are the symmetric, positive definite matrices of density and thermal capacity products and thermal conductivities, respectively, $\mathbf{j}_w$ is the vector of dual facet heat currents, $\mathbf{G}$ is a discrete primal grid gradient matrix, $\tilde{\mathbf{S}}$ is the discrete dual grid divergence operator, and $\mathbf{q}_w(t)$ is the source vector of ohmic dual cell losses. With the primal-dual grid relation $\tilde{\mathbf{S}} = -\mathbf{G}$ [7], ODE system (7) can be reformulated into the pH-DAE system (2) with the dof vector $\mathbf{x} = [\mathbf{T}; \mathbf{f}_w]$ and

(9)
$$\mathbf{E} = \begin{bmatrix} \mathbf{M}_{c\rho} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{G}^\top\mathbf{M}_\lambda \\ -\mathbf{M}_\lambda\mathbf{G} & \mathbf{0} \end{bmatrix}, \mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_\lambda \end{bmatrix}$$

and $\mathbf{B}$ as before. Note that a coupled electro-thermal pH-DAE formulation needs to consider thermal volume losses within the source vector $\mathbf{q}_w(t)$.

The full paper will give details on the coupled pH-DAE systems formulations.

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